reflects the Fourier stiffness distribution profile. (A convergence trend is exhibited because the higher-order harmonics of the solution diminish in magnitude.) The total deflection obtained by summing the Fourier contributions is plotted along the plane of symmetry ($\theta = 0$–$180°$) in Fig. 5. Also shown are the deflections obtained for axisymmetric shells with uniform stiffness distributions corresponding to the magnitude of the stiff and weak sides. As expected, these results bracket the solution for the variable-rigidity shell.

The same problem was a run with only three and five terms retained in the stiffness series expansions. Variation in the third harmonic of deflection (w_2) along the meridian is plotted in Fig. 6 for the cases where K is set equal to 3, 5, and 10. The higher harmonic coupling effects appear to diminish as more terms are retained in the solution (i.e., as K increases). For economic purposes, the number of integration points used in obtaining the preceding numerical results was $N = 26$.

The basic character of the solutions is illustrated for the relatively coarse grid. Additional studies have shown increased accuracy when a finer finite-difference mesh is taken; this should be considered when the results presented in this paper are interpreted.

References

[1] Budiansky, B. and Radkowski, P. P., "Numerical Analysis of Unsymmetrical Bending of Shells of Revolution," *AIAA Journal*, Vol. 1, No. 8, Aug. 1963, pp. 1833–1842.

[2] Kalnins, A., "Analysis of Shells of Revolution Subjected to Symmetrical and Nonsymmetrical Loads," *Journal of Applied Mechanics*, Sept. 1964.

[3] Cohen, G. A., "Computer Analysis of Asymmetrical Deformation of Orthotropic Shells of Revolution," *AIAA Journal*, Vol. 2, No. 5, May 1964, pp. 932–934.

[4] Sanders, J. L., Jr., "An Improved First-Approximation Theory for Thin Shells," Rept. 24, June 1959, NASA.

[5] Potters, M. L., "A Matrix Method for the Solution of a Second Order Difference Equation in Two Variables," Rept. MR 19, 1955, Mathematich Centrum, Amsterdam, The Netherlands.

[6] Cappelli, A. P., Nishimoto, T. S., and Pauley, K. E., "Fourier Approach to the Analysis of a Class of Unsymmetrical Shells Including Shear Distortion," Structures TR 139, Dec. 1965, Space Div. of North American Rockwell Corp., Downey, Calif.

[7] Greenbaum, G. A., "Comments on 'Numerical Analysis of Unsymmetrical Bending of Shells of Revolution,'" *AIAA Journal*, Vol. 2, No. 3, March 1964, pp. 590–591.

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Finite-Element Postbuckling Analysis of Thin Elastic Plates

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A finite-element formulation for determining postbuckling response of thin elastic plates is presented. The method employs an iterative approach to arrive at equilibrium configurations. For each iterate, increments in displacements are estimated by formulating an approximate incremental stiffness matrix and solving the linearized incremental equilibrium equations. The approximate incremental stiffness is determined by combining a geometric stiffness with the stiffness of the unstressed, but displaced, structure. A method of deriving consistent geometric stiffness matrices for two-dimensional plate elements is presented and the principal terms in this matrix are evaluated for a triangular element. Results of the determination of postbuckling response for some typical plates are given.

Nomenclature

$[B]$	= matrix of influence functions specifying increments in element strains for increments in element nodal displacements
$C_{ijkl};[\bar{C}]$	= linear elastic moduli; plane stress elastic constitutive matrix
$[\tilde{D}]$	= matrix of influence functions specifying increments in element middle surface displacements for increments in element nodal displacements
$E_{ij};\Delta E_{ij}$	= Green's strain tensor and its increment
h	= plate thickness
i,j,k,l	= indices with range of three
K_I,K_G,K_E	= incremental, geometric, and element stiffnesses, respectively
$M_{\alpha\beta},N_{\alpha\beta}$	= in-plane stress couples and stress resultants
o	= subscript indicating original configuration
Q_α	= shear stress resultants
$\{r_E\}$	= vector of element nodal displacements, Eq. (13)
$\{\Delta r\};\{\Delta R\}$	= structure nodal displacement increments and nodal force increments
$S_{ij};\Delta S_{ij}$	= Kirchhoff's stress tensor and its increment
$S_0;dS_0$	= surface area and differential surface area
$T_i;\Delta T_i$	= Kirchhoff stress vector and its increment
$[T]$	= corner transformation matrix
T	= superscript indicating transpose
$\bar{u}_i;u_i$	= initial displacements and displacement increments referred to local coordinate system, Fig. 1
$\bar{U}_i;U_i$	= initial displacements and displacement increments referred to global coordinate system, Fig. 1
$\{u\},\{v\},\{w\}$	= vectors of element displacement increments, Fig. 2, in local coordinate system
$\{U\},\{V\},\{W\}$	= vectors of element displacement increments in global coordinate system
x,y,z	= local coordinate system, Fig. 1
X,Y,Z	= global coordinate system, Fig. 1
α,β	= indices with range of two
δ	= indicates virtual variation

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$\{\phi_u\};\{\phi_v\};\{\phi_w\}$ = vectors of interpolating functions specifying increments of element displacements associated with nodal increments $\{u\}$, $\{v\}$, $\{w\}$, respectively
$\Gamma,\Delta\Gamma$ = designation of structure configuration and its increment, Fig. 1
$\sim$ = indicates field variable to distinguish from nodal value
, = indicates partial differentiation
$\{\ \},[\ ]$ = vector and matrix symbols

I. Introduction

MANY recent developments have taken place in the field of "finite-element" or "discrete element" analysis of large displacement and stability problems. Mallett and Marcal[1] have given a summary of these developments and presented a systematic formulation for nonlinear problems.

The approach adopted in this work differs from that of Ref. 1 in that it utilizes a set of moving coordinate systems. The analysis is based upon the formulation of an approximate incremental assembled stiffness matrix $[K_I]$, which is used to estimate increments in nodal displacements $\{\Delta r\}$, for a given load increment $\{\Delta R\}$, by solving the equation

$$[K_I]\{\Delta r\} = \{\Delta R\} \tag{1}$$

The solution procedure employed in the analysis has been described in detail elsewhere.[2] The procedure is iterative as well as incremental. The concept may be simply stated as follows. For any deformed configuration the difference between the sum of the element equilibrating forces and the applied nodal force, at any node, is treated as an "unbalanced" nodal force. The unbalanced nodal forces may be reduced to arbitrarily small quantities by successive corrections to the configuration. The configuration may therefore be made to approach the equilibrium configuration to within any arbitrarily chosen limit of a metric on the unbalanced forces.

The advantages of this approach are that the assembled stiffness matrix need not be exact and that the solution error may be controlled. Geometric nonlinearities in the equilibrium equations are readily incorporated in the assembly procedure and geometric nonlinearities associated with the strain-displacement equations are incorporated into a transformation of element deformations.[2] Material nonlinearities may be included, but are not dealt with here. The limiting feature is the ability of the analyst to predict the element equilibrating forces for any deformed configuration.

Although most of the formulation of this paper is independent of the element employed, examples were solved using a triangular element with 15 degrees of freedom. Membrane displacement functions are those associated with the constant strain triangle[3] and bending displacement functions are those associated with the Hsieh-Clough-Tocher triangle.[4]

II. Formulation of Incremental Equilibrium Equations

Incremental equilibrium equations will be derived for the restricted class of large deflection problems where engineering strains remain small, with respect to one, but displacement gradients are otherwise unrestricted. Under these conditions the product of an increment in Green's strain tensor[5] (ΔE_{ij}) and Kirchhoff's stress tensor[5] (S_{ij}) represents work. Since these tensors are referred to the original configuration of the structure, integration may be carried out over this original configuration. Coordinate systems are rectangular Cartesian coordinate systems and the principle of virtual displacements is used throughout.

Referring to Fig. 1, let Γ represent the deformed equilibrium configuration of the structure under the loads R (the "initial" configuration), and $\Gamma + \Delta\Gamma$ represent the deformed equilibrium configuration under the loads $R + \Delta R$ (the "incremented" configuration). Let $\bar{U}_i$ be the displacements in configuration Γ and $\bar{U}_i + U_i$ the displacements in configuration $\Gamma + \Delta\Gamma$ with respect to the global coordinate system X_i. The quantities U_i therefore represent displacement increments due to $\Delta\Gamma$.

For configuration Γ the stresses and strains are

$$2E_{ij} = \bar{U}_{i,j} + \bar{U}_{j,i} + \bar{U}_{k,i}\bar{U}_{k,j} \tag{2}$$

and

$$S_{ij} = C_{ijkl}E_{kl} \tag{3}$$

The present analysis deals with elastic response and therefore C_{ijkl} are regarded as constant. More sophisticated constitutive relationships may be used without changing the approach.

For configuration $\Gamma + \Delta\Gamma$, the stresses and strains are

$$2(E_{ij} + \Delta E_{ij}) = (\bar{U}_i + U_i)_{,j} + (\bar{U}_j + U_j)_{,i} + (\bar{U}_k + U_k)_{,i}(\bar{U}_k + U_k)_{,j} \tag{4}$$

and

$$S_{ij} + \Delta S_{ij} = C_{ijkl}(E_{kl} + \Delta E_{kl}) \tag{5}$$

Subtraction of Eq. (2) from Eq. (4) yields the explicit expression for ΔE_{ij},

$$\Delta E_{ij} = \tfrac{1}{2}\{U_{i,j} + U_{j,i} + U_{k,i}U_{k,j} + \bar{U}_{k,i}U_{k,j} + U_{k,i}\bar{U}_{k,j}\} \tag{6}$$

To develop the incremental equilibrium equation we follow the procedure of Biot[6] and take the difference between the equilibrium equations for positions $\Gamma + \Delta\Gamma$ and Γ.

Applying the principle of virtual displacements in position $\Gamma + \Delta\Gamma$, and regarding the displacement increments U_i as the variables, yields

$$\int_{V_0} (S_{ij} + \Delta S_{ij})\delta(\Delta E_{ij})_{\Gamma+\Delta\Gamma} dV_0 = \int_{S_0} (T_i + \Delta T_i)\delta U_i dS_0 \tag{7}$$

where V_0 is the original volume, T_i and ΔT_i are the specified Kirchhoff stress vector, and its increment, acting on the surface of the structure, and ΔE_{ij} is given by Eq. (6). The equilibrium equation in configuration Γ may be obtained by letting ΔT_i, ΔS_{ij}, and U_i approach zero. The result is

$$\int_{V_0} S_{ij}\delta(\Delta E_{ij})_{\Gamma} dV_0 = \int_{S_0} T_i \delta U_i dS_0 \tag{8}$$

Subtracting Eq. (8) from Eq. (7) yields the incremental equilibrium equation for position $\Gamma + \Delta\Gamma$,

$$\frac{1}{2}\int_{V_0} S_{ij}\delta(U_{k,i}U_{k,j})dV_0 + \int_{V_0} \Delta S_{ij}\delta(\Delta E_{ij})_{\Gamma+\Delta\Gamma} dV_0 = \int_{S_0} \Delta T_i \delta U_i dS_0 \tag{9}$$

The first term in Eq. (9) represents the work of the "initial" stress state on the virtual variations of increments in displacement gradients and is of the same order of magnitude as the second term. It gives rise to the so-called "geometric stiffness." The second term in Eq. (9) represents the work of the increments in the stress state on the virtual variations of increments in strain. The strain increment, however, contains terms dependent on the displacement gradients present in the "initial" configuration as indicated in Eq. (6). The term on the right-hand side represents the work done by increments in generalized external loading, ΔR, on virtual variations of displacement increments.

Although it is possible to deal with Eq. (9) directly, it is more conveniently evaluated by subdividing the structure into subregions (elements) and summing the values, after integration in each subregion. For this purpose a different local coordinate system may be used in each subregion.

Referring to Fig. 1, let $\bar{u}_i$ be the displacements in configuration Γ and $\bar{u}_i + u_i$ be the displacements in configuration $\Gamma + \Delta\Gamma$, with respect to a local coordinate system x_i. The displacements $\bar{u}_i$ and displacement increments u_i are referenced to the original shape of the element, oriented with respect to coordinates x_i, which are determined by configuration Γ. If engineering strains remain small, a valid description of the state of stress and strain in the element is obtained by rewriting Eqs. (2–6) with $\bar{u}_i$ and u_i replacing $\bar{U}_i$ and U_i. Since the work equations [(7) and (8)] remain valid regardless of the coordinate system used to express the variables in the integrands, Eq. (9) may be rewritten as

$$\sum \frac{1}{2}\int_{V_0} S_{ij}\delta(u_{k,i}u_{k,j})dV_0 + \sum \int_{V_0} \Delta S_{ij}\delta(\Delta E_{ij})_{\Gamma+\Delta\Gamma}dV_0 = \sum \int_{S_0} \Delta T_i \delta U_i dS_0 \quad (10)$$

providing all displacement quantities are subjected to compatible arbitrary variations.

The strain increment expression in Eq. (10) is now

$$\Delta E_{ij} = \tfrac{1}{2}\{u_{i,j} + u_{j,i} + u_{k,i}u_{k,j} + \bar{u}_{k,i}u_{k,j} + u_{k,i}\bar{u}_{k,j}\} \quad (11)$$

However, an important distinction now exists between Eq. (11) and Eq. (6). For small engineering strains, the product terms in Eq. (11) may be reduced to higher-order quantities by selecting a sufficiently fine subdivision of the structure. This, in general, is not true for Eq. (6).

In utilizing the finite-element approach and Eq. (10), the coordinate system x_i is considered to be a fixed local coordinate system for each element that is determined by the current geometric configuration Γ. The integrals are evaluated over each element and summed. This is accomplished by formulating element stiffnesses with respect to the local coordinate systems, transforming to the global coordinate system, according to the current configuration, and assembling by the direct stiffness procedure.

For each element, the first term of Eq. (10) gives rise to what will be called the "element geometric stiffness" matrix. (Similar matrices have been designated as "geometric stiffness" matrices by Argyris,[7] "initial stress" matrices by Martin,[8] and "stability matrices" by Hartz.[9]) The second term of Eq. (10) gives rise to the "element stiffness" of an unstressed element in the deformed configuration. Since out-of-plane displacement gradients may be made arbitrarily small by reducing the size of the mesh, the product terms in Eq. (11) may be dropped in accordance with small deflection plate theory. The stiffness matrix arising from this term therefore yields uncoupled stiffness matrices identical to those associated with the membrane and bending behavior of the element.

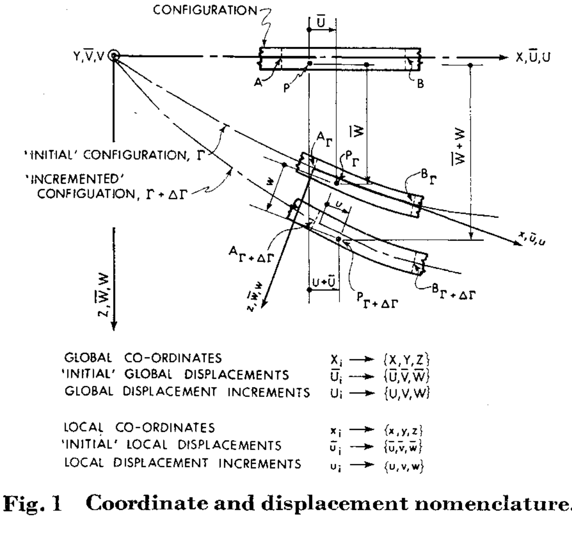

Fig. 1 Coordinate and displacement nomenclature.

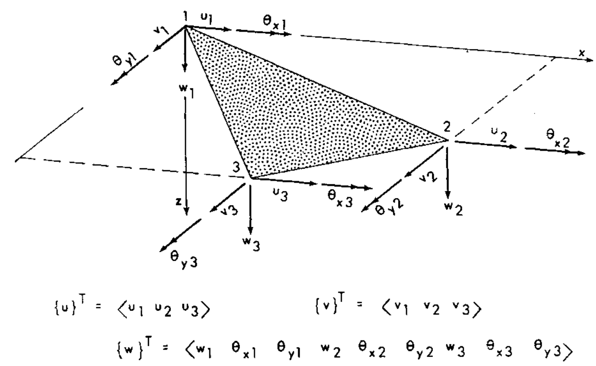

Fig. 2 Nodal displacements and nodal vectors.

III. A General Development of the Element Geometric Stiffness Matrix for Plate Elements

The displacement† field for any plate element may be expressed, in the local coordinate system, in terms of nodal displacements, by utilizing a set of shape functions and the Kirchhoff assumptions. This results in the equation‡

$$\begin{Bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{Bmatrix} = \begin{bmatrix} \{\phi_u\}^T & \cdots & -z\{\phi_{w,x}\}^T \\ \cdots & \{\phi_v\}^T & -z\{\phi_{w,y}\}^T \\ \cdots & \cdots & \{\phi_w\}^T \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{v\} \\ \{w\} \end{Bmatrix} \quad (12)$$

$$= [\tilde{D}]\{r_E\} \quad (13)$$

where $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ are displacements in the coordinate directions x, y, and z; $\{\phi_u\}$, $\{\phi_v\}$, and $\{\phi_w\}$ are vectors of interpolating functions associated with the nodal displacement vectors $\{u\}$, $\{v\}$, and $\{w\}$, respectively; and $[\tilde{D}]$ and $\{r_E\}$ are defined by identification with the corresponding terms of Eq. (12). Generalized coordinates have been dispensed with since they are unnecessary if the development follows the procedures of Felippa.[10]

For the triangular element utilized in this analysis the nodal vectors are shown in Fig. 2. Corresponding interpolation functions may be found in Refs. 10 or 13.

The element geometric stiffness matrix arises from the expression

$$\frac{1}{2}\int_{V_0} \tilde{S}_{ij}\delta(\tilde{u}_{k,i}\tilde{u}_{k,j})dV_0 \quad (14)$$

which appears in Eq. (10). This expression may now be put in matrix form as

$$\frac{1}{2}\int_{V_0} \tilde{S}_{ij}\delta\left[\langle \tilde{u}\ \tilde{v}\ \tilde{w}\rangle_{,i} \begin{Bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{Bmatrix}_{,j}\right] dV_0 = \frac{1}{2}\,\delta\{r_E\}^T\left[\int_{V_0} [\tilde{D}_{,i}]^T\tilde{S}_{ij}[\tilde{D}_{,j}]dV_0\right]\{r_E\} \quad (15)$$

The integral in expression (15) represents the sum of nine separate matrix products, one associated with each stress component, which is a symmetric matrix because of the symmetry of S_{ij} and the summation. Taking the variation with respect to the nodal displacements therefore yields

$$[K_G] = \int_{V_0} [\tilde{D}_{,i}]^T\tilde{S}_{ij}[\tilde{D}_{,j}]dV_0 \quad (16)$$

where $[K_G]$ is the geometric stiffness matrix. Substituting for

† In the following the term displacement is to be understood to refer to displacement increment. The word increment is omitted for the sake of brevity.

‡ Field variables, except for interpolating functions, will be designated by a tilde (~) to distinguish them from nodal values.

$$[K_G] = \int_A \left[\begin{array}{c:c:c} \{\Phi_{u,\alpha}\} N_{\alpha\beta} \{\Phi_{u,\beta}\}^T & & \{\Phi_{u,\alpha}\} M_{\alpha\beta} \{\Phi_{w,x\beta}\}^T \\ & & -\{\Phi_{u,\alpha}\} Q_\alpha \{\Phi_{w,x}\}^T \\ \hdashline & \{\Phi_{v,\alpha}\} N_{\alpha\beta} \{\Phi_{v,\beta}\}^T, & \{\Phi_{v,\alpha}\} M_{\alpha\beta} \{\Phi_{w,y\beta}\}^T \\ & & -\{\Phi_{v,\alpha}\} Q_\alpha \{\Phi_{w,y}\}^T \\ \hdashline \{\Phi_{w,x\alpha}\} M_{\alpha\beta} \{\Phi_{u,\beta}\}^T, & \{\Phi_{w,y\alpha}\} M_{\alpha\beta} \{\Phi_{v,\beta}\}^T, & \{\Phi_{w,\alpha}\} N_{\alpha\beta} \{\Phi_{w,\beta}\}^T + \Psi_1 \\ -\{\Phi_{w,x}\} Q_\alpha \{\Phi_{u,\alpha}\}^T, & -\{\Phi_{w,y}\} Q_\alpha \{\Phi_{v,\alpha}\}^T & \end{array}\right] dA$$

where $\Psi_1 = N_{\alpha\beta} \frac{h^2}{12} \left[\{\Phi_{w,x\alpha}\}\{\Phi_{w,x\beta}\}^T + \{\Phi_{w,y\alpha}\}\{\Phi_{w,y\beta}\}^T \right]$

Fig. 3 Complete geometric stiffness matrix for two-dimensional elements.

$[\tilde{D}]$, from Eq. (12), integrating through the thickness, and using the normal assumptions of plate theory with respect to distribution of stresses, yields the definitive form of $[K_G]$, shown in Fig. 3, in terms of stress couples and stress resultants. The stress resultants and stress couples are illustrated in Fig. 4, and defined by the relationship

$$\langle \tilde{N}_{\alpha\beta}\, \tilde{Q}_\alpha \rangle = \int_{-h/2}^{h/2} \langle \tilde{S}_{\alpha\beta}\, \tilde{S}_{\alpha z} \rangle dz \qquad (17)$$

and

$$\tilde{M}_{\alpha\beta} = \int_{-h/2}^{h/2} - z\tilde{S}_{\alpha\beta}\, dz \qquad (18)$$

where h is the thickness of the plate and Greek indices have a range of two. The matrix in Fig. 3 is "complete" in the sense that it indicates the effect of membrane forces on bending stiffness and the effect of stress couples on membrane stiffness.

For small displacement increments the only term in the geometric stiffness matrix whose elements are of the same order of magnitude as the elements of the element stiffness matrix is the term in the lower right-hand corner. The effective element geometric stiffness may therefore be evaluated by approximating $[K_G]$ as

$$[K_G] \doteq \int_A \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \{\phi_{w,\alpha}\} \tilde{N}_{\alpha\beta} \{\phi_{w,\beta}\}^T \end{bmatrix} dA \qquad (19)$$

This geometric stiffness corresponds to the terms normally included in the plate formulation for the influence of membrane forces on bending stiffness.[11] Equation (19) yields terms associated with a "consistent" geometric stiffness in the same sense as consistent mass matrices.[12] The evaluation of this matrix has been carried out for the Hsieh-Clough-Tocher triangular element following the methods of Felippa[10] and has been incorporated into the solution procedure. Details of this evaluation may be found in Ref. 13.

IV. Element Stiffness, Assembly, and Solution Procedure

The element stiffness matrix in the local coordinate system is determined by evaluating the second term of Eq. (10) in terms of nodal displacements. The strain increments in a two-dimensional element may be evaluated by utilizing Eq. (12). The result is

$$\begin{Bmatrix} \tilde{\epsilon}_x \\ \tilde{\epsilon}_y \\ \tilde{\gamma}_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \tilde{u}}{\partial x} \\ \dfrac{\partial \tilde{v}}{\partial y} \\ \dfrac{\partial \tilde{u}}{\partial y} + \dfrac{\partial \tilde{v}}{\partial x} \end{Bmatrix}$$

$$= \begin{bmatrix} \{\phi_{u,x}\}^T & \cdots & -z\{\phi_{w,xx}\}^T \\ \cdots & \{\phi_{v,y}\}^T & -z\{\phi_{w,yy}\}^T \\ \{\phi_{u,y}\}^T & \{\phi_{v,x}\}^T & -2z\{\phi_{w,xy}\}^T \end{bmatrix} \begin{Bmatrix} \{u\} \\ \{v\} \\ \{w\} \end{Bmatrix} \qquad (20)$$

$$= [\tilde{B}]\{r_E\} \qquad (21)$$

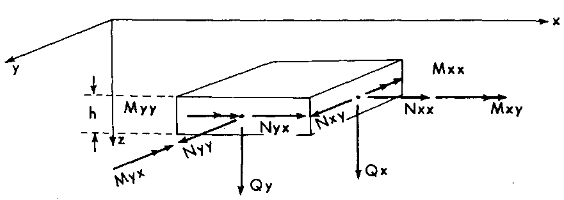

Fig. 4 Stress couples and resultants.

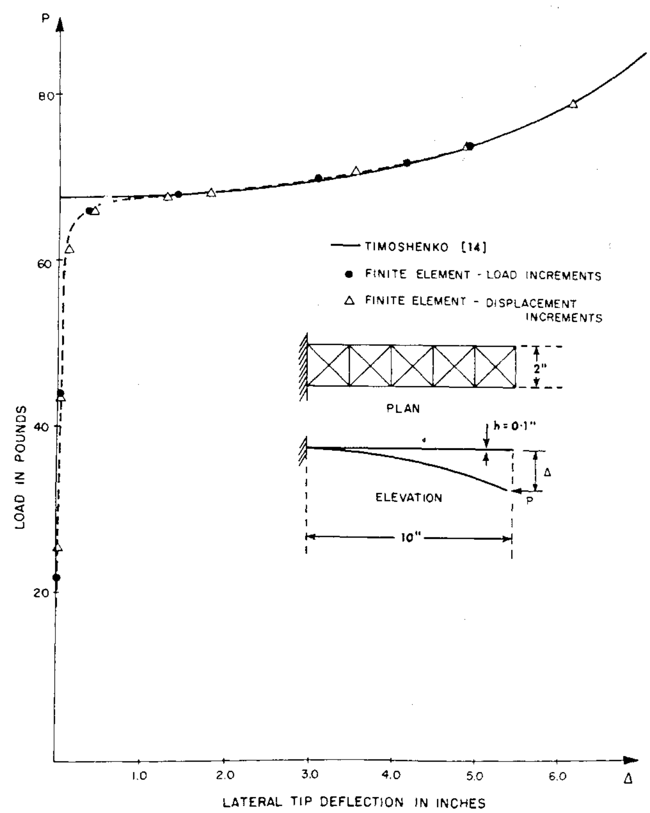

Fig. 5 Postbuckling load deflection of cantilevered plate.

where $\tilde{\epsilon}_x$, $\tilde{\epsilon}_y$, and $\tilde{\gamma}_{xy}$ are increments in engineering strains and $[\tilde{B}]$ is defined by identification with the corresponding term of Eq. (20). Product terms in the strain increment-displacement increment relationships have been dropped for the reasons previously stated.

Evaluation of the element stiffness matrix is carried out by performing the integration in the equation[2]

$$[K_E] = \int_{V_0} [\tilde{B}]^T[\tilde{C}][\tilde{B}]dV_0 \tag{22}$$

where $[\tilde{B}]$ is defined by Eqs. (20) and (21) and $[\tilde{C}]$ is the two-dimensional constitutive relationship.

For elastic behavior and the small deflection relationship of Eq. (20), this matrix is composed of the uncoupled membrane and bending stiffness matrices, designated as $[K_P]$ and $[K_B]$, respectively;

$$[K_E] = \begin{bmatrix} [K_P] & \cdots \\ \cdots & [K_B] \end{bmatrix} \tag{23}$$

For the element displacement patterns of this analysis $[K_P]$ is derived in Ref. 3, and $[K_B]$ in Ref. 4.

The incremental element stiffness, for element k, may now be written in the global coordinate system as

$$[K]_k = [T]_k{}^T[K_I][T]_k \tag{24}$$

where

$$[K_I]_k = [K_G]_k + [K_E]_k \tag{25}$$

and $[T]_k$ is the displacement transformation matrix arising in the infinitesimal displacement transformation

$$\begin{Bmatrix} \{u\} \\ \{v\} \\ \{w\} \end{Bmatrix}_k = [T]_k \begin{Bmatrix} \{U\} \\ \{V\} \\ \{W\} \end{Bmatrix}_k \tag{26}$$

This transformation couples the membrane and bending behavior. The incremental equilibrium equation for the entire structure may then be written in the form of Eq. (1), where $[K_I]$ is the incremental stiffness matrix obtained by direct stiffness assembly procedures.

For a given increment in load, Eq. (1) is solved for increments of nodal displacements $\{\Delta r\}$. Nodal displacements are then incremented, element equilibrating forces are evaluated, and the difference between the sum of the equilibrating forces and the applied loads is considered as the load increment vector for the next iterate. A new stiffness matrix is evaluated for the current state of stress and geometry and the solution is repeated until the unbalanced nodal forces are arbitrarily small. Details of this procedure are given in Ref. 2.

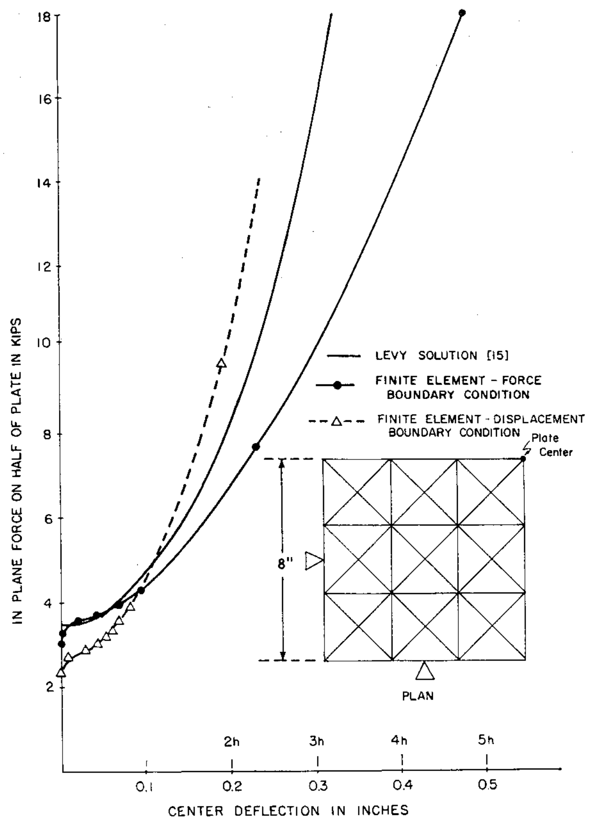

Fig. 6 Postbuckling load deflection of square plate.

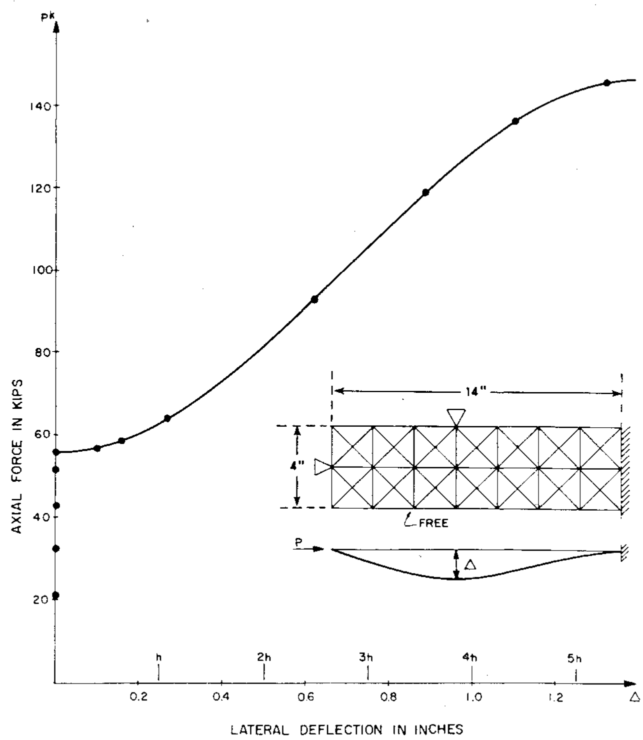

Fig. 7 Postbuckling load deflection of a flange plate.

V. Results

The analysis was programed for a CDC 6400 computer and applied to a variety of thin elastic plate problems in an

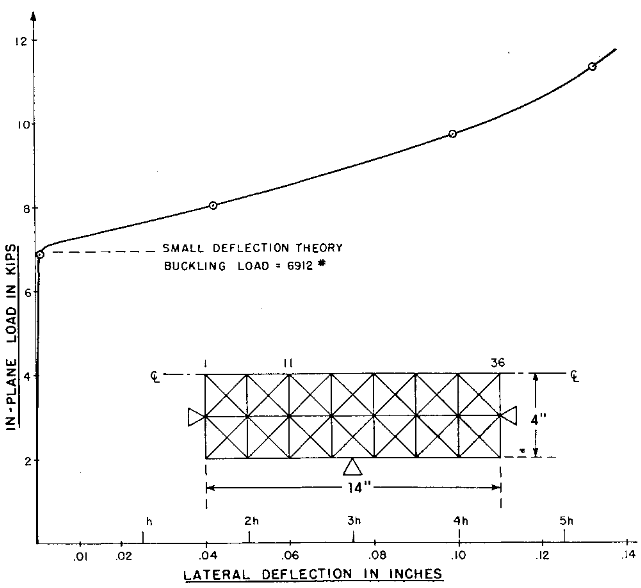

Fig. 8 Postbuckling load deflection of simply supported plate with aspect ratio 1.75.

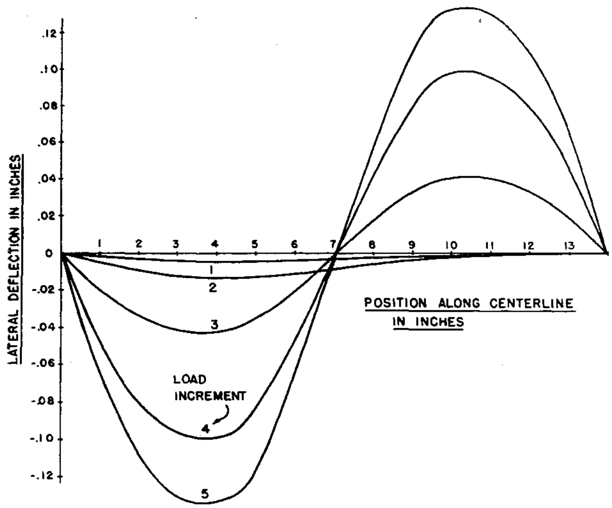

Fig. 9 Centerline profile of simply supported rectangular plate with aspect ratio 1.75.

attempt to predict postbuckling behavior. Where possible, results are compared to published results in the literature. Further details are available in Ref. 13.

Figure 5 compares the tip deflection for the "elastica" problem with the theoretical solution.[14] This problem was solved without the use of $[K_G]$ in Ref. 2. Figure 6 compares the centre deflection of a simply supported square plate with the results of Levy.[15] The force boundary condition and displacement boundary condition effectively bracket Levy's intermediate boundary condition.

To illustrate the versatility of the method, Fig. 7 shows the deflection of a point on the free edge of a simulated flange plate, with fixed, free, and simply supported edges. Figures 8 and 9 show the deflection of a point on the centerline of a rectangular simply supported plate (aspect ratio of 1.75) and the centerline configuration at various load levels, respectively.

In producing these results a single small constant concentrated load, normal to the original plane of the plate, was maintained on the structure throughout the analysis. The postbuckling behavior then arose naturally by incrementing displacement boundary conditions in the plane of the plate.

VI. Summary and Conclusions

A method of analysis has been presented for application to problems involving postbuckling behavior of elastic plates. The method is based on assembling an approximate incremental stiffness matrix for the structure which is used to predict increments in nodal displacements. The solution procedure is iterative and based on an equilibrium balance between reactive structure forces and applied loads. The limiting feature of the analysis is the ability to predict equilibrating element forces for a given element configuration. It may be concluded, from the results presented, that the method may form the basis of a reasonable approach for investigating problems in the postbuckling range.

References

[1] Mallett, R. H. and Marcal, P. V., "Finite-Element Analysis of Nonlinear Structures," *Journal of the Structural Division, Proceedings of the American Society of Civil Engineers*, Vol. 94, No. ST9, Sept. 1968, pp. 2081–2105.

[2] Murray, D. W. and Wilson, E. L., "Finite-Element Large Deflection Analysis of Plates," *Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers*, Vol. 95, No. EM1, Feb. 1969, pp. 143–165.

[3] Wilson, E. L., "Finite-Element Analysis of Two-Dimensional Structures," SESM Rept. 63-2, 1963, Univ. of California, Berkeley, Calif.

[4] Clough, R. W. and Tocher, J. L., "Finite-Element Stiffness Matrices for Analysis of Plate Bending," TR-66-80, 1966, Air Force Flight Dynamics Lab., pp. 515–545.

[5] Fung, Y. C., *Foundations of Solid Mechanics*, Prentice-Hall, Englewood Cliffs, N.J., 1965, pp. 91, 436–439.

[6] Biot, M. A., *Mechanics of Incremental Deformations*, Wiley, New York, 1965.

[7] Argyris, J. H., Kelsey, S., and Kamel, H., "Matrix Methods of Structural Analysis," *Agardograph 72*, edited by F. de Veubeke, Pergamon Press, New York, 1964, pp. 1–164.

[8] Martin, H. C., "On the Derivation of Stiffness Matrices for the Analysis of Large Deflection and Stability Problems," TR-66-80, 1966, Air Force Flight Dynamics Lab., pp. 697–716.

[9] Hartz, B. J., "Matrix Formulation of Structural Stability Problems," *Journal of the Structural Division, Proceedings of the American Society of Civil Engineers*, Vol. 91, No. ST6, 1965, pp. 141–157.

[10] Felippa, C. A., "Refined Finite Element Analysis of Linear and Non-Linear Two-Dimensional Structures," SESM Rept. 66-22, Univ. of California, Berkeley, Calif., 1966.

[11] Gallagher, R. H. and Paldlog, J., "Discrete Element Approach to Structural Instability Analysis," *AIAA Journal*, Vol. 1, No. 6, June 1963, pp. 1437–1439.

[12] Archer, J. S., "Consistent Mass Matrix for Distributed Mass Systems," *Journal of the Structural Division, Proceedings of the American Society of Civil Engineers*, Vol. 89, No. ST4, 1963, pp. 161–178.

[13] Murray, D. W., "Large Deflection Analysis of Plates," SESM Rept. 67-44, Univ. of California, Berkeley, Calif., 1967.

[14] Timoshenko, S. P. and Gere, J. M., *Theory of Elastic Stability*, 2nd ed., McGraw-Hill, New York, 1961.

[15] Levy, S., "Bending of Rectangular Plates with Large Deflections," TN 846, 1942, NACA.